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A simple model of competition between teams ☆

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Abstract

We model competition between two teams (that may differ in size) as an all-pay contest with incomplete information where team members exert effort to increase the performance of their own team. The team with the higher performance wins, and its members enjoy the prize as a public good. The value of the prize is identical to members of the same team but is unknown to those of the other team. We focus on the case in which a team's performance is the sum of its individual members' performances and analyze monotone equilibria in which members of the same team exert the same effort. We find that the bigger team is more likely to win if individual performance is a concave function of effort, less likely if convex, and equally likely if linear. We also provide a complete characterization of the equilibria for the case in which individual performance is a power function of effort and team value is uniformly distributed. For this case we also investigate how probabilities of winning, total team performance and individual payoffs are affected by the size of the team. The results shed light on the "group-size paradox".

JEL classification: C72; D72; D74; D82

Keywords: Team contests; Group size paradox

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1. Introduction

Many economic, political and social activities are performed by groups or organizations rather than individuals. When firms compete, the strategic interaction is essentially between *collectives* of individuals. Electoral competition between candidates involves strategic interaction between *teams* consisting of the candidates themselves, their consultants and the activists that support them. Similarly, R&D races are carried out between *labs* of scientists. Lobbying efforts are carried out by interest *groups* who need to coordinate the actions of their members in response to the actions of other interest groups. Likewise, ethnic conflicts involve different *peoples* who are united by a common background, such as religion or ethnic origin.

Despite the ubiquity of strategic interactions between groups, the majority of economic analyses treat players in game theoretic models as individual entities. While this may be a convenient simplification, it ignores the interplay between *intra*-group strategizing (i.e. how each member of a group reasons about the actions of other members of the *same* group) and *inter*-group strategizing (how each member of a group reasons about the actions of the members of the *opposing* groups), which may have important implications for the outcome of the interaction. Furthermore, explicitly modeling each participating unit as a collective of decision makers may produce insights on how group attributes (such as, for instance, size) can affect outcomes.

We therefore propose and analyze a model of competition between groups. We focus on the case of two competing groups, or teams, of possibly different sizes. We model the interaction between them as a generalized all-pay auction with incomplete information. The members of each team individually exert effort to increase the performance of their team via an aggregator called *team performance function*, and the team with higher performance wins. Each player individually bears the cost of his own effort regardless of whether his team wins.

Since we assume that individuals act non-cooperatively — each maximizing his personal payoff — a natural question that arises is what makes a collection of individuals into a "team"? We propose to view a team as a cohesive set of individuals who share the same values, which are commonly known to them. We therefore assume that when a team wins, its members enjoy the prize as a *public* good. This assumption is motivated by many of the prevalent examples of team competition. When, for example, a gun-control group, LGBT rights movement or an environmental group lobbies for a change in legislation, their success is shared by all the members as a group. Similarly, when a sports teams wins a championship, it is often the case that its members each assign equal importance to winning. Indeed, many models in the group contests literature assume that the value of winning is a public good.[1]

Clearly, different teams may assign different values to winning. A loss for one group may mean that it ceases to exist, while for another group it may only be a minor setback. For example, a lab or small start-up may have to close down if it does not win an R&D contest, whereas a large corporation will still have other projects. Similarly, defeat in a political competition may mean retirement for the team of one candidate, whereas the team of another candidate will try again in the future. The consequences of introducing new environmental regulation will be different for an industry group that lobbied against it than for an environmental group that lobbied for it. It is often the case that each competing team faces some uncertainty about the other team's value of winning. A team may have some beliefs, or some noisy information about their rival's value of winning, but it is unlikely they will know it with certainty. We therefore assume that each team

[1] Some of the most notable papers include Katz et al. (1990); Baik (1993, 2008); Baik et al. (2001); Barbieri et al. (2014); Chowdhury et al. (2013); and Kolmar and Rommeswinkel (2013).

privately draws its value of winning from some distribution. For tractability, we assume that the draws are made independently from the same distribution.

Each member of a team must decide how much effort to exert. Since effort is costly, each team member faces a trade-off between the cost of his own effort and the effect it will have on the likelihood of winning. Thus, an important feature of a contest is how the individual efforts of a team's members aggregate into the total performance of the team. In many situations, a team member's effort spills over, thus making his teammates more productive. For example, in combat, a breakthrough of the enemy frontline by one squad makes the breakthrough by other squads more likely. In these situations there are complementarities between team members' efforts. In other situations, complementarities may not come into play. For example, in campaign fund-raising, the amount raised in one district does not significantly influence the amount raised in another. Likewise, the success of one product line may not have an effect on a company's other product lines. In some lobbying activities, each lobbyist on a team targets different politicians, each working independently.

In this paper, we focus on the case in which there are no complementarities between the performances of the team members. In particular, we assume that team performance is additively separable in individual performances. In addition to its tractability, the model is also the typical case analyzed by the majority of group contest models (which we discuss in more detail below).[2] On the other hand, we recognize the importance of complementarities and provide a more general analysis in the Appendix, which allows for within-team complementarities.

We analyze the pure strategy monotone Bayesian Nash equilibria in which members of the same team exert the same effort. We are mainly interested in understanding the effect of team size on the equilibrium probability of winning and on the players' equilibrium payoffs. This relates to the well-known "group-size paradox", which argues that free-riding puts a bigger group at a disadvantage (see Olson (1965)). We show that this phenomenon depends on how individual effort translates into individual performance. In particular, the bigger team is more likely to win if the individual performance function is concave, less likely if it is convex and equally likely if it is linear. The underlying intuition is that the curvature of the individual performance function determines whether *in equilibrium* additional members effectively augment or reduce the marginal productivity of existing members. A concave individual performance function is characteristic of physical tasks that typically exhibit decreasing marginal return on effort, whereas a convex individual performance function captures "cognitive" tasks such as research or innovation where marginal return on effort may be increasing. We also show that if the individual performance function is concave or linear, then the expected payoff to a member of the bigger team is higher than that to a member of the smaller team.

The non-linearity of the individual performance function makes it difficult to obtain a closed-form expression for the players' strategies in our model (by analogy, in standard "winners-pay" auctions with risk-averse bidders there is *no* explicit characterization of the equilibrium bid in first or second-price auctions for *general* preferences). Fortunately, for the class of individual performance functions given by e^{α}, where e is an individual's effort and $\alpha > 0$, we are able to provide a closed-form solution when each team draws its value from a uniform distribution (by analogy, in winners-pay auctions with risk aversion there is a closed-form solution for the case of absolute risk aversion).

2 Some of the most notable papers include Katz et al. (1990), Katz and Tokatlidu (1996), Esteban and Ray (2001), Baik et al. (2001) and Nitzan and Ueda (2011).

This setup allows us to present a number of novel insights. First, we provide a simple closed-form expression for each team's equilibrium probability of winning. This expression clearly shows that the probability of winning changes with team size and α, i.e., the parameter of the curvature of the individual performance function. Second, we provide a simple closed-form expression for each team's equilibrium performance, which allows us to characterize how team performance changes with team sizes and α. Third, we identify a threshold $\hat{\alpha}$, which depends on team sizes, such that the equilibrium expected payoff to a member of the bigger team is higher than that to a member of the smaller team if and only if the curvature parameter α is below that threshold. In addition, we show how the equilibrium payoffs change with team sizes and α. These observations allow us to specify whether a new individual will join the bigger or the smaller team and when each team finds it profitable to recruit a new member. To the best of our knowledge, this is the first paper to establish these comparative statics.

The remainder of the paper is organized as follows: Section 2 discusses the related literature. In Section 3, we present the model and in Section 4 the main results. Section 5 analyzes the case in which individual performance takes the form of a power function. Section 6 concludes. In the Appendix, we discuss a couple of supplementary results.

2. Related literature

The paper is closely related to the literature on contest theory. Most papers in this literature can be classified according to the following binary categorizations:

1. Who is competing: individuals or teams?
2. How is the winner chosen: stochastically or deterministically?
3. Information structure: complete or incomplete?

The literature can thus be organized into a $2 \times 2 \times 2$ design:

INDIVIDUALS	Complete	Incomplete
Stochastic		
Deterministic		

TEAMS	Complete	Incomplete
Stochastic		
Deterministic		

There is already a vast literature that fills in the cells in the left half of the table. Some of the prominent papers in the complete information column include Hillman and Riley (1989), Baye et al. (1996) and Siegel (2009) for the "deterministic" case and Siegel (2009) and Cornes and Hartley (2005) for the "stochastic" case . The incomplete information column includes Amann and Leininger (1996), Lizzeri and Persico (2000), Kirkegaard (2013) and Siegel (2014) for the "deterministic" case and Ryvkin (2010), Ewerhart and Quartieri (2013) and Ewerhart (2014) for the "stochastic" case.

There is also an extensive literature on team contests with complete information. This literature includes Skaperdas (1998), Nitzan (1991), Esteban and Ray (2001, 2008), Nitzan and Ueda (2009, 2011), Münster (2007, 2009), Konrad and Leininger (2007), and Konrad and Kovenock (2009), among many others. In particular, our assumption that the value of winning is a public good among team members follows Baik et al. (2001), Topolyan (2013), Chowdhury and Topolyan (2015) and Chowdhury et al. (2016). They assume that team performance is either the minimum or the maximum of the individual performances, whereas we focus on additively separable (possibly non-linear) functions that aggregate individual efforts into a total team performance.

The current model falls into the incomplete information column, which has only been filled in recently by Fu et al. (2015) and Barbieri and Malueg (2016). The former analyzes a general model that accommodates each of the cells on the right-hand side of the table. However, their model differs from ours in that they study a multi-battle contest: Players from two equally sized teams form pairwise matches to compete in distinct two-player all-pay auctions, and a team wins if and only if its players win a majority of the auctions. In contrast, we analyze a contest between teams of different sizes in which the members of both teams participate simultaneously in one big all-pay auction.

The paper by Barbieri and Malueg (2016) is more closely related to our model since it also analyzes a static incomplete information all-pay auction between teams that may differ in size. However, they assume that the reward for winning has an independent private value for each team member and that the team's performance is equal to the maximal performance among its members. Under this specification they show that in the case of two teams with different distributions of private values, a team's probability of winning increases (decreases) with size if its distribution is inelastic (elastic). We assume that all members of a team have the same commonly known value of winning, but that it is unknown to the opposing team. We link the size advantage/disadvantage to the *shape* of the individual performance function. In addition, we also analyze the effect of group size on individual welfare and show that it also depends on the shape of the individual performance function.

The effect of group size on the probability of winning has been the subject of a number of important works in the literature. The seminal game-theoretic paper on this topic is Esteban and Ray (2001) (ER henceforth). They analyze a complete information "Tullock" contest between two teams (i.e., a stochastic contest in which the probability of winning is proportional to the team's performance). Team performance is linear in individual effort, though not necessarily an individual's cost of effort. ER also assumes that the prize can have a private-good component.

The two main results of ER in the pure public good case are related to the results obtained here. First, they show that if the cost function is convex, then (1) the bigger team is more likely to win, and (2) if only one of the teams increases in size, then that team's probability of winning increases. Second, they show that a player's equilibrium payoff increases with his team's size. The key feature of ER's results is that a *convex cost function* favors the bigger team.

In our model, the cost function is linear but the individual performance function is not.[3] We find that a *concave individual performance function* favors the bigger team. It is not difficult to see that a convex cost function in ER implies essentially the same preference as a concave individual performance function in our model.[4] Hence, ER's results agree with our finding.

Despite the resemblance between some of the results, the model presented here differs substantially from ER. In terms of modeling, we study an all-pay auction (vs. a Tullock contest in ER) between two teams with incomplete information (vs. complete information in ER) of the opposing team's value of winning. Thus, the two frameworks are fundamentally different and in fact belong to two relatively independent branches in the contest literature. The ER framework fits situations in which each team knows *exactly* the winning value of its rival, and there is exogenous noise in the contest outcome such that the side putting in the most effort may not necessarily win (furthermore, the likelihood of winning takes a very particular form, which is

[3] In our model, it is more natural to impose the non-linearity on the production function rather than the cost function since we obtain unambiguous results for the concave case and a concave *production* function is more readily interpretable than a concave *cost* function.

[4] We thank one of the referees for pointing this out.

crucial to the analysis). However, many of the examples mentioned in the Introduction typically have the feature that each competitor is uncertain about the value his rival attaches to winning. In addition, it is often the case that the competitor with the highest output wins (as in the case of an R&D competition, or in an election or lobbying competition in which the total number of votes count). For these environments, our model appears to offer a better fit.

The model presented here also offers a broader scope of results than ER. First, we show that in the case of an additive separable team performance function, if the individual performance function is convex (which corresponds to a concave cost function in ER), then the smaller team has an advantage. This is in contrast to ER, in which the effect of group size is indeterminate when the cost function is concave. Second, for the class of individual power performance functions, we present new comparative static results with respect to the curvature of the individual performance function and the sizes of the teams. Furthermore, we analyze the question (not examined by ER) of which team a new member would want to join and which team would want to recruit him.[5]

Following ER, a number of subsequent papers also investigated the effect of group size on the likelihood of winning. Nitzan and Ueda (2009, 2011) explore the effect of adding another stage to the ER model. Nitzan and Ueda (2009) add a second stage in which every member of the winning group decides (simultaneously) how much to use of the public good prize where the level chosen by one member has an externality on other members. Thus, when choosing their effort levels, team members take into account the implications in the second stage. The authors show that if the infimum (w.r.t. a player's choice of usage) of the elasticity of a player's benefit from the prize w.r.t. team size is higher (lower) than the supremum of the elasticity of the marginal cost of effort, then the smaller (bigger) team is more likely to win. In contrast, Nitzan and Ueda (2011) add a preliminary stage before the levels of efforts are chosen, in which the prize-sharing rule is determined by a benevolent and forward-looking team leader. They show that this endogenous determination of group-sharing rules *completely eliminates* the group-size paradox, in which a bigger group always has a higher winning probability than a smaller group. A different source for the group-size paradox is explored by Barbieri et al. (2014) who show that when two teams compete in an all-pay auction with *complete* information the bigger team is *less* likely to win when team output is determined by the *highest individual effort*. Thus, the source of group size effect in these models is very different than in the current model. Furthermore, they only discuss the effect of group size on the likelihood of winning, as opposed to the model presented here which also analyzes the effect on payoffs.

3. The model

Two teams, B (for "big") and S (for "small"), compete for a prize. Team B has n_B players and team S has n_S players, where $n_B > n_S$. We denote by X a generic team and by Y the opposing team. Within team X members are indexed by $1, 2, ..., n_X$. Competition takes the following form. All players simultaneously choose some action from $\mathbb{R}_+$. Player i's chosen action e_i is interpreted as the amount of effort that player i exerts.

Team X's overall performance in the competition is given by the *team performance function:*

$$H_{n_X}(e_1, ..., e_{n_X}) = \sum_{i=1}^{n_X} h(e_i)$$

[5] There is also a fundamental difference in the mathematics of the two results, which we illustrate in the online appendix.

where h is the *individual performance function*, which is strictly increasing, twice differentiable and satisfies $h(0) = 0$. As mentioned in the Introduction, additively separable team performance functions are commonly used in the contest literature and capture environments where there are no complementarities between players on a team.[6] Note that the team performance function only depends on a team's size but none of its other attributes since we are mainly interested in the effect of team size on the competition's outcome.

The team with the higher performance wins the prize. A tie is broken by the toss of a fair coin. Every member of team X receives a payoff of $v_X \in [0, 1]$ if the team wins. v_X is known to members of team X before the competition starts, but is unknown to members of the other team. It is common knowledge that v_B and v_S are both drawn from the same distribution F, where F admits a strictly positive and continuous density function f. Regardless of which team wins the prize, each player pays a cost equal to the amount of effort he exerts. Thus, the net payoff to player i on team X who exerted effort e_i is $\mathbf{1}_X v_X - e_i$, where $\mathbf{1}_X$ is equal to 1 if team X wins and 0 otherwise.

4. Analysis

This section analyzes the equilibria of the game and explores the effect of team sizes on the equilibrium outcome. We focus on pure strategy Bayesian Nash equilibria[7] (PBNE) that satisfy the following properties:

1. Monotonicity: Each player's effort is weakly increasing in his valuation of the prize.
2. In-team symmetry: Players on the same team use the same strategy.

Both conditions provide tractability. Moreover, under in-team symmetry the size effect can be cleanly differentiated from other factors that affect a team's performance since within the same team players are strategically homogeneous.[8]

An in-team symmetric monotone PBNE can be represented as (e_B, e_S) where e_X is the effort function of all members of team X. We further denote the equilibrium performance of team X given valuation v as $P_X(v) \equiv \sum_{i=1}^{n_X} h(e_X(v)) = n_X h(e_X(v))$. Clearly P_X is also weakly increasing in v. In addition, denote the inverse of the equilibrium team performance as $P_X^{-1}(x) \equiv \sup_v P_X(v) \leq x$. Finally, let G_X denote the ex ante equilibrium distribution of P_X.

The following are properties of in-team symmetric monotone PBNE, which will prove useful later on.

Lemma 1. *In any in-team symmetric monotone PBNE:*

1. $P_B(1) = P_S(1) \equiv \overline{P}$.

[6] Apart from their popularity in the literature, we focus on additively separable functions also for their expositional clarity. In the Appendix, we provide an analysis of a more general family of team performance functions which can accommodate complementarities. Some results in the main text can be reformulated as implications of that more general analysis.

[7] One of the reasons we chose to study the incomplete information model, rather than its complete information counterpart, is that pure strategy equilibria typically do not exist under complete information. On the other hand, since we impose no additional assumptions on the valuation distribution F, all the results hold for a model with "almost complete" information in which F is arbitrarily close to a degenerate distribution.

[8] We note that in-team *asymmetric* equilibria may exist. On the other hand, as we shall see, under certain conditions on h, all nontrivial monotone PBNE are in-team symmetric.

2. *G_X is continuous and strictly increasing over* $[0, \overline{P}]$.
3. $\min\{G_B(0), G_S(0)\} = 0$.
4. *P_X is everywhere differentiable on* $(P_X^{-1}(0), 1)$ *and P_X^{-1} is everywhere differentiable on* $(0, \overline{P})$.

Proof. Parts 1–3 are implied by the monotonicity assumption and the extension of Lemmas 1, 2, 4 and 5 in Amann and Leininger (1996) to our setting.[9]

To show differentiability,[10] choose any v where $P \equiv P_X(v) \in (0, \overline{P})$. Given valuation v, it is optimal for a player on team X to exert $e_X(v)$ instead of an effort that decreases team performance by any $\delta > 0$. Thus we have:

$$v \int_0^{P_Y^{-1}(P)} f(w)dw - e_X(v) \geq v \int_0^{P_Y^{-1}(P-\delta)} f(w)dw - e_X(v) + k(\delta, e_X(v))$$

where $k(\delta, e)$ is implicitly defined by the equation $h(e) - h\Big(e - k(\delta, e)\Big) = \delta$. It is straightforward to verify that $\lim_{\delta \searrow 0} \frac{\delta}{k(\delta,e)} = \frac{1}{(h^{-1})'(h(e))}$, which is finite and positive for $e > 0$. It follows after rearranging the inequality that $1 \leq \frac{v}{k(\delta,e_X(v))} \int_{P_Y^{-1}(P-\delta)}^{P_Y^{-1}(P)} f(w)dw$, which in turn implies:

$$1 \leq \frac{vf(v)}{(h^{-1})'(h(e_X(v)))} \limsup_{\delta \searrow 0} \Big[\frac{P_Y^{-1}(P) - P_Y^{-1}(P-\delta)}{\delta}\Big]. \tag{4.1}$$

For any small $\delta > 0$ let $v(\delta) > 0$ be a valuation such that $P_X(v(\delta)) = P - \delta$. Clearly, $\lim_{\delta \searrow 0} v(\delta) = v$. Given the valuation $v(\delta)$, it is optimal for a player on team X to exert $e_X(v(\delta))$ rather than an effort that increases team performance by δ. Thus we have

$$v(\delta) \int_0^{P_Y^{-1}(P-\delta)} f(w)dw - e_X(v(\delta)) \geq v(\delta) \int_0^{P_Y^{-1}(P)} f(w)dw - e_X(v(\delta)) - d(\delta, e_X(v(\delta)))$$

where $d(\delta, e)$ is implicitly defined by the equation $h(e + d(\delta, e)) - h(e) = \delta$. It is straightforward to verify that $\lim_{\delta \searrow 0} \frac{\delta}{d(\delta,e)} = \frac{1}{(h^{-1})'(h(e))}$, which is finite and positive for $e > 0$. Using an argument similar to the derivation of inequality (4.1) we have

$$1 \geq \frac{vf(v)}{(h^{-1})'(h(e_X(v)))} \liminf_{\delta \searrow 0} \Big[\frac{P_Y^{-1}(P) - P_Y^{-1}(P-\delta)}{\delta}\Big]. \tag{4.2}$$

By a symmetric argument, inequalities analogous to (4.1) and (4.2) hold for δ approaching 0 from below. The inequalities together imply that P_Y^{-1} is differentiable at P and moreover

[9] Lemmas 1, 2, 4 and 5 in Amann and Leininger (1996) relate to an incomplete information all-pay auction between two individuals. These results can be straightforwardly extended to our setting with essentially the same proofs. On the other hand, Lemma 3 of Amann and Leininger (1996) shows that in their setting the PBNE must be monotone, which does not extend to our setting. Instead we impose monotonicity by assumption.

[10] The argument we use is based on the proofs of Lemma 7 of Lizzeri and Persico (2000) and Lemma 1 of Lu and Parreiras (2017). We thank one of the referees for bringing this argument to our attention.

$(P_Y^{-1})'(P) = \frac{(h^{-1})'(h(e_X(v)))}{vf(v)} > 0$. Similarly P_X^{-1} is also differentiable and has strictly positive derivative over $(0, \overline{P})$. Part 4 follows. □

We now establish the uniqueness of in-team symmetric monotone PBNE.

Lemma 2. *There is a unique in-team symmetric monotone PBNE.*

Proof. Fix an in-team symmetric monotone PBNE (e_B, e_S). For player i on team X with valuation $v > 0$, if $e_X(v) > 0$ then optimality implies the first-order condition $f\Big(P_Y^{-1}\Big(n_X h(e_X(v))\Big)\Big) \times (P_Y^{-1})'\Big(n_X h(e_X(v))\Big)h'(e_X(v))v = 1$.

Substituting in $P_X(v) = n_X h(e_X(v))$ and using change of variable $x = P_X(v)$ the first-order condition simplifies to:

$$f(P_Y^{-1}(x))(P_Y^{-1})'(x)h'(h^{-1}(x/n_X))P_X^{-1}(x) = 1. \tag{4.3}$$

Note that $(P_Y^{-1})'(x)$ is defined because P_Y^{-1} is differentiable at $x = P_X(v) > 0$ by Lemma 1. It follows from optimality and Lemma 1 that (P_B^{-1}, P_S^{-1}) correspond to an in-team symmetric monotone PBNE if and only if it solves the following differential equations:

$$f(P_S^{-1}(x))(P_S^{-1})'(x)h'(h^{-1}(x/n_B))P_B^{-1}(x) = 1$$
$$f(P_B^{-1}(x))(P_B^{-1})'(x)h'(h^{-1}(x/n_S))P_S^{-1}(x) = 1$$

subject to the boundary conditions $P_B(1) = P_S(1)$ and $\min\{P_B^{-1}(0), P_S^{-1}(0)\} = 0$.

These conditions are identical to those that characterize the monotone PBNE effort functions (P_B, P_S) in an all-pay contest between two players B and S whose valuations are independently distributed according to F, such that the performance of player X is the same as his chosen amount of effort, and the cost of exerting effort e is equal to $c_X(e) \equiv \int_0^e \frac{1}{h'(h^{-1}(s/n_X))}ds$. By the analysis in Section 4.1 of Kirkegaard (2013), the auxiliary two-player game (called game Γ_2 there) has a unique monotone PBNE.[11] This establishes the existence and uniqueness of an in-team symmetric monotone PBNE of our game. □

Our first result establishes that the *bigger* team is *more* (*less*) likely to win if there are *diminishing* (*increasing*) returns on effort for players. Formally, the effect of team size on the probability of winning is determined by the curvature of h. This means that the bigger team has an advantage in situations where the largest contribution to performance occurs early on. On the other hand, in tasks where greater expertise (which increases the rate of return) requires greater effort, the smaller team will have an advantage. Thus, one advantage of the model is that we can identify a *qualitative* feature of the environment — the nature of the task at hand — which predicts which team is more likely to win.

[11] Kirkegaard (2013) imposes additional constraints on c_X, but they are irrelevant for the existence and uniqueness of the solution to the boundary value problem. Kirkegaard (2013) also assumes that the lowest valuation is strictly positive in order to ensure that the differential equations are everywhere Lipschitz. His method of proof (tracing the inverse bidding functions starting from the upper bound $\overline{P} = P_B(1) = P_S(1)$ and then verifying that the boundary condition $\min\{P_B^{-1}(0), P_S^{-1}(0)\} = 0$ is satisfied) nonetheless holds in our case where Lipschitz is violated only at 0.

Theorem 1. *In the in-team symmetric monotone PBNE:*

1. *If h is strictly concave, then $P_B(v) > P_S(v)$ for $v \in (0, 1)$ and hence team B is more likely to win.*
2. *If h is strictly convex, then $P_B(v) < P_S(v)$ for $v \in (0, 1)$ and hence team S is more likely to win.*
3. *If h is linear, then $P_B(v) = P_S(v)$ for $v \in [0, 1]$ and hence both teams win with the same probability.*

Proof. Fix any v such that $P_B(v) = P_S(v) > 0$. Part 1 of Lemma 1 implies that such v exists. Define $t \equiv P_B(v) = P_S(v)$. By first order condition (4.3), we have:

$$f(v)(P_S^{-1})'(t)h'(h^{-1}(t/n_B))v = 1$$
$$f(v)(P_B^{-1})'(t)h'(h^{-1}(t/n_S))v = 1.$$

It follows that:

$$\frac{h'(h^{-1}(t/n_B))}{h'(h^{-1}(t/n_S))} = \frac{(P_B^{-1})'(t)}{(P_S^{-1})'(t)}. \tag{4.4}$$

Note that for $n > 0$ we have $\frac{d}{dn}h'(h^{-1}(t/n)) = -h''(h^{-1}(t/n))(h^{-1})'(t/n)tn^{-2}$. Since h is strictly increasing, $(h^{-1})'(t/n)tn^{-2} > 0$. If h is strictly concave then the derivative is strictly positive, which implies the left hand side of equation (4.4) is greater than 1. It follows that $(P_B^{-1})'(t) > (P_S^{-1})'(t)$ or equivalently $P_B'(v) < P_S'(v)$. Thus, whenever the curves of P_B and P_S intersect when P_B and P_S are strictly positive, the curve of P_S is steeper than the curve of P_B, implying that they intersect at most once in the strictly positive region since part 2 of Lemma 1 implies that P_B and P_S are continuous. Part 1 of Lemma 1 implies that they intersect at $\overline{P}$ (when $v = 1$). Thus, $P_B(v) > P_S(v)$ for any $v < 1$ where $P_B(v) > 0$. This, combined with part 3 of Lemma 1, implies $P_B(v) > P_S(v)$ for any $v \in (0, 1)$. Therefore, team B is more likely to win. Part 2 is proven with a symmetric argument.

If h is linear, then h' is a constant, implying that the boundary value problem characterizing the in-team symmetric monotone PBNE given in the proof of Lemma 2 does not depend on team size, which in turn, along with uniqueness and symmetry, implies that $P_B = P_S$. Part 3 follows. □

Theorem 1 implies that the existence of the "group-size paradox" depends on whether a player's marginal return on effort is increasing or decreasing. To see the intuition of this result, consider the case of a strictly concave h. If a team with n members incurs a total cost of effort C, then the team's performance will be $nh(C/n)$. Taking the team as a whole, the marginal return on effort is therefore $\frac{d}{dC}nh(C/n) = h'(C/n)$. Since h is strictly concave, it follows that for a given C, the marginal return on the effort of a team increases with team size. Thus, given the same total cost C, we have $n_B h(C/n_B) > n_S h(C/n_S)$, implying that the bigger team has higher performance.

Moreover, for an n-member team to achieve a total performance of T, each member's effort must be $h^{-1}(T/n)$, and therefore each member's individual marginal return on effort is $h'(h^{-1}(T/n))$. This individual marginal return on effort is also increasing in n. In other words, adding members makes existing members more "productive". Thus, to achieve the same team performance everyone on the bigger team can now reduce his effort, which simultaneously increases his marginal return on effort.

Remark 1. Although the analysis focuses on in-team symmetric equilibria, in all three cases — concave, convex or linear h — the loss of generality from requiring in-team symmetry is limited. In particular:

- If h is strictly concave, then the in-team symmetric equilibrium is the only monotone PBNE.
- If h is strictly convex, then the in-team symmetric equilibrium is the only monotone PBNE in which there does not exist a player who *always* exerts zero effort.[12]
- If h is linear, then there are a continuum of monotone PBNE, all of which yield the same equilibrium team performances (P_B, P_S) as the in-team symmetric equilibrium.

These results are formally stated and proved in the Appendix.

Would a player prefer to be on the bigger team or the smaller team? To answer this question it is not sufficient to merely compare the winning probabilities since a higher winning probability may require a higher level of individual effort, which may offset the gain from a higher winning probability. Suppose h is either concave, convex or linear and that the players coordinate on the unique in-team symmetric monotone PBNE.[13] Clearly, the ex ante equilibrium expected payoff for a player on team X depends only on the number of players on each team. Let $u(n_X, n_Y)$ denote this expected payoff for a player on team X. The following proposition shows that if h is weakly concave, that is, if marginal return on effort is weakly diminishing, then in equilibrium members of the bigger team are better off than members of the smaller one.

Proposition 1. *If h is weakly concave, then $u(n_B, n_S) > u(n_S, n_B)$.*

Proof. It is clear that[14]

$$u(n_X, n_Y) = \int_0^1 F\left(P_Y^{-1}(P_X(v))\right) v dF(v) - \int_0^1 e_X(v) dF(v).$$

First, suppose that h is strictly concave. Thus, by Theorem 1(1), $P_B(v) \geq P_S(v)$ with the inequality being strict for a set of values with positive measure under F. Suppose a player on team B unilaterally deviates to the following effort function[15]:

$$\hat{e}(v) = \max\left\{0, h^{-1}\left(P_S(v) - (n_B - 1)h(e_B(v))\right)\right\}.$$

In other words, he shirks as much as he can in order to ensure that for any v the resulting team performance $\hat{P}_B(v)$ is as high as $P_S(v)$. The fact that $P_B(v) \geq P_S(v)$ implies that $\hat{e}(v) \leq e_B(v)$. It follows from $P_B(v) \geq \hat{P}_B(v) \geq P_S(v)$ that

$$P_S^{-1}(\hat{P}_B(v)) \geq \hat{P}_B^{-1}(P_S(v)) \geq P_B^{-1}(P_S(v))$$

[12] It seems only natural to assume that every player exerts some effort on some occasions, since if a player shirks all the time, then he has no impact on the competition and is essentially absent from the team. The "effective" team size should not include him.

[13] We obtain in-team symmetry for free by Remark 1 if h is strictly concave.

[14] Note that this expression ignores ties, which by Lemma 1 occur with zero probability.

[15] Here we assume h is also defined on the negative domain, although a player cannot choose a negative effort. Monotonicity and $h(0) = 0$ imply $h(e) < 0$ for $e < 0$.

with at least one of the two above inequalities being strict for any v such that $P_B(v) > P_S(v)$. Thus, $P_B(v) > P_S(v)$ implies $P_S^{-1}(\hat{P}_B(v)) > P_B^{-1}(P_S(v))$

We now show that $\hat{e}(v) \leq e_S(v)$. This is clearly true for any v such that $e_B(v) \leq e_S(v)$. For any v such that $e_B(v) \geq e_S(v)$, we have $(n_B - 1)h(e_B(v)) \geq n_S h(e_B(v)) \geq n_S h(e_S(v)) = P_S(v)$, implying $h^{-1}\Big(P_S(v) - (n_B - 1)h(e_B(v))\Big) < 0$, which in turn implies $\hat{e}(v) = 0 \leq e_S(v)$.

Thus,

$$\begin{aligned} u(n_B, n_S) &\geq \int_0^1 F\Big(P_S^{-1}(\hat{P}_B(v))\Big) v dF(v) - \int_0^1 \hat{e}(v) dF(v) \\ &> \int_0^1 F\Big(P_B^{-1}(P_S(v))\Big) v dF(v) - \int_0^1 \hat{e}(v) dF(v) \\ &\geq \int_0^1 F\Big(P_B^{-1}(P_S(v))\Big) v dF(v) - \int_0^1 e_S(v) dF(v) \\ &= u(n_S, n_B). \end{aligned}$$

If h is linear then $P_B = P_S$ by Theorem 1(3). Thus, $v = P_S^{-1}(P_B(v)) = P_B^{-1}(P_S(v))$ and $e_B(v) \leq e_S(v)$ with the inequality being strict if $e_B(v) > 0$. Thus,

$$\begin{aligned} u(n_B, n_S) &= \int_0^1 F(v) v dF(v) - \int_0^1 e_B(v) dF(v) \\ &> \int_0^1 F(v) v dF(v) - \int_0^1 e_S(v) dF(v) \\ &= u(n_S, n_B). \quad \square \end{aligned}$$

If h is convex, then the argument used in the proof of Proposition 1 does not hold and the welfare comparison in general is unclear.[16] However, we can establish that for *some* convex h, it is still the case that each member of the bigger team receives a higher expected payoff than each member of the smaller team, even though the smaller team is more likely to win.

Proposition 2. *For any $n_B > n_S$, there exists some convex h such that $u(n_B, n_S) > u(n_S, n_B)$.*

Proof. Pick any n_B and n_S where $n_B > n_S$. Let $U(\alpha|n_X, n_Y)$ be equal to $u(n_X, n_Y)$ for the game with $h(e) = e^\alpha$ where $\alpha > 0$. Clearly, $U(\alpha|n_X, n_Y)$ is continuous in α. Since $U(\alpha|n_B, n_S) > U(\alpha|n_S, n_B)$ for any $\alpha \leq 1$ by Proposition 1, there exists some $\hat{\alpha} > 1$ (in which case h is strictly convex) such that $U(\hat{\alpha}|n_B, n_S) > U(\hat{\alpha}|n_S, n_B)$. $\square$

In the next section, we zoom in on the class of individual performance functions used in the above proof, namely those in which $h(e) = e^\alpha$. In this case, we are able to derive a precise

[16] Note that the proof relied on $P_B(v) \geq P_S(v)$, which is not the case for a convex h.

threshold of α (which depends on the teams' sizes) for which the bigger team's payoff is higher even though it is less likely to win.

5. Power performance functions

The non-linearity of the individual performance function h prevents us from obtaining closed-form expressions for team performance, winning probabilities and payoffs in the in-team symmetric monotone PBNE. Consequently, it is difficult to conduct comparative statics on these key variables with respect to the size of each team and the curvature of the individual performance function. In light of this, we focus on a one-parameter class of additively separable team performance functions in which individual performance takes the form of the power function $h(e) = e^{\alpha}$, where $\alpha > 0$. To further simplify the analysis, we assume that team values are uniformly distributed on [0, 1]. All the results in this section assume these specifications of h and F (the distribution of teams' values), and so we omit the assumptions when stating the results.

The first set of results provide a closed-form expression for the equilibrium performance of each team and show how it changes with respect to a team's own size and that of the opposing team.

Proposition 3. *(Team performance) In the in-team symmetric monotone PBNE:*

1. The total performance of a team with value v is given by:

$$P_B(v) = \Big(\frac{\beta\sigma}{\beta+\sigma}\Big)^{\alpha} v^{\alpha(1+\frac{\sigma}{\beta})}$$

$$P_S(v) = \Big(\frac{\beta\sigma}{\beta+\sigma}\Big)^{\alpha} v^{\alpha(1+\frac{\beta}{\sigma})}$$

where $\beta = n_B^{\frac{1}{\alpha}-1}$ *and* $\sigma = n_S^{\frac{1}{\alpha}-1}$.

2. A team's total performance increases (decreases) with its own size for any valuation if $\alpha < 1$ *(*$\alpha > 1$*).*

3. There exists a threshold $\overline{v}(\alpha, n_B, n_S) \in (0, 1)$ *such that if* $\alpha < 1$ *(*$\alpha > 1$*), then a team's total performance increases (decreases) with the size of the opposing team if* $v > \overline{v}(\alpha, n_B, n_S)$ *and decreases (increases) if* $v < \overline{v}(\alpha, n_B, n_S)$.

Proof. It is straightforward (yet tedious) to prove part 1 by verifying that the given functions solve the boundary value problem characterizing the in-team symmetric monotone PBNE in the proof of Lemma 2.

By part 1, team X's performance in the in-team symmetric monotone PBNE is $P_X(v|k, n_Y) = n_Y^{1-\alpha} k^{-\alpha} v^{\alpha k}$ where $k = 1 + (n_Y/n_X)^{1/\alpha - 1}$. For convenience, suppose that k is a continuous variable. Thus

$$\frac{\partial P_X}{\partial k} = n_Y^{1-\alpha} k^{-\alpha} v^{\alpha k} (\ln v - \frac{\alpha}{k}) < 0.$$

Clearly k is decreasing in n_X if $\alpha < 1$ or increasing if $\alpha > 1$. Thus, P_X is increasing in n_X if $\alpha < 1$ or decreasing if $\alpha > 1$, thereby establishing part 2.

For part 3, we have:

$$\frac{dP_X}{dn_Y} = \frac{\partial P_X}{\partial k}\frac{\partial k}{\partial n_Y} + \frac{\partial P_X}{\partial n_Y} = (1-\alpha) k^{-\alpha} v^{\alpha k} n_Y^{-\alpha} \Big[\frac{1}{\alpha}(\ln v - \frac{\alpha}{k})\Big(\frac{n_Y}{n_X}\Big)^{1/\alpha - 1} + 1\Big].$$

The term in the square brackets is positive if $v > \overline{v} \equiv e^{\alpha[1/k-(n_X/n_Y)^{1/\alpha-1}]}$ or negative if the inequality is reversed. Since

$$1/k - (n_X/n_Y)^{1/\alpha-1} = n_X^{1/\alpha-1}\Big(\frac{1}{n_X^{1/\alpha-1}+n_Y^{1/\alpha-1}} - \frac{1}{n_Y^{1/\alpha-1}}\Big) < 0$$

for any $n_X, n_Y > 0$, we conclude that $\overline{v} \in (0, 1)$. Thus, if $\alpha < 1$ (> 1), then $\mathrm{d}P_X/\mathrm{d}n_Y$ is positive (negative) for $v > \overline{v}$ or negative (positive) for $v < \overline{v}$, thereby establishing part 3. □

The second set of results are concerned with equilibrium winning probabilities. We first show that these take a very simple form, which enables straightforward comparative statics with respect to each team's size and the curvature parameter α.

Proposition 4. *(Winning probability) In the in-team symmetric monotone PBNE:*

1. The winning probability of team X is equal to:

$$\frac{n_X^{\frac{1}{\alpha}-1}}{n_X^{\frac{1}{\alpha}-1}+n_Y^{\frac{1}{\alpha}-1}}.$$

2. If $\alpha < 1$ ($\alpha > 1$), then a team's winning probability increases (decreases) with its own size and decreases (increases) with the opposing team's size.

3. As the individual performance function becomes more convex, i.e., as α increases, the winning probability of the bigger (smaller) team decreases (increases).

Proof. For part 1, note that in the in-team symmetric monotone PBNE the winning probability of team X is

$$\int_0^1 P_Y^{-1}\Big(P_X(v)\Big)dv = \int_0^1 v^{(n_Y/n_X)^{\frac{1}{\alpha}-1}}dv = \frac{n_X^{\frac{1}{\alpha}-1}}{n_X^{\frac{1}{\alpha}-1}+n_Y^{\frac{1}{\alpha}-1}}.$$

If $\alpha < 1$ (> 1), this probability is clearly increasing (decreasing) in n_X and decreasing (increasing) in n_Y, thus establishing part 2. In addition, the right-hand side is decreasing in α if $n_X > n_Y$ and increasing in α if $n_X < n_Y$, which proves part 3. □

The third set of results focuses on players' equilibrium payoffs. We first present a closed-form expression of a player's ex ante expected payoff in the in-team symmetric monotone PBNE and then characterize how equilibrium payoffs are affected by changes in each team's size and in α. Finally, we show that there exists a threshold $\hat{\alpha} > 1$ (which depends on teams sizes) for the curvature parameter α, such that the payoff to a member of the *bigger* team is *higher* than that to a member of the smaller team if and only if α is *below* that threshold. In other words, there exists a range of values of α for which the smaller team is more likely to win, but the payoff to each of its members is lower than that for the bigger team.

Proposition 5. *(Payoffs) In the in-team symmetric monotone PBNE:*

1. The ex ante payoff to a member of team X is equal to:

$$\pi(r, n_X) = \left(1 - \frac{1}{n_X} \cdot \frac{1}{1+r}\right) \cdot \frac{r}{2r+1}$$

where $r = (n_X/n_Y)^{1/\alpha - 1}$.

2. As the individual performance function becomes more convex — i.e., as α increases — the equilibrium payoff to a member of the bigger (smaller) team decreases (increases).

3. There exists $\hat{\alpha}(n_B, n_S) > 1$ *such that the payoff to a member of the bigger team is higher than that to a member of the smaller team if and only if* $\alpha < \hat{\alpha}(n_B, n_S)$.

4. If $\alpha < 1$*, a player's payoff increases with the size of his own team and decreases with the size of the opposing team. If* $\alpha = 1$*, a player's payoff increases with the size of his own team and remains constant with the size of the opposing team. If* $\alpha > 1$*, a player's payoff increases with the size of the opposing team; moreover, it decreases (increases) with the size of his own team if its size is greater (less) than a constant* $\overline{n}(\alpha)$ $(\underline{n}(\alpha))$.

Proof. For part 1, note that the equilibrium payoff to a member on team X is equal to:

$$\begin{aligned}\pi_X(\alpha) &\equiv \int_0^1 \left[P_Y^{-1}\Big(P_X(v)\Big)v - h^{-1}\Big(P_X(v)/n_X\Big)\right]dv \\ &= \left(1 - \frac{n_Y^{\frac{1}{\alpha}-1}}{n_X^{\frac{1}{\alpha}-1} + n_Y^{\frac{1}{\alpha}-1}} \frac{1}{n_X}\right) \frac{n_X^{\frac{1}{\alpha}-1}}{2n_X^{\frac{1}{\alpha}-1} + n_Y^{\frac{1}{\alpha}-1}}.\end{aligned}$$

It is straightforward to verify that $\pi_B(\alpha)$ is decreasing in α and $\pi_S(\alpha)$ is increasing in α, which establishes part 2.

To show part 3, note that $\lim_{\alpha\to 0} \pi_B(\alpha) = 1/2 > 0 = \lim_{\alpha\to 0} \pi_S(\alpha)$ and $\lim_{\alpha\to\infty} \pi_B(\alpha) = \left(1 - \frac{1}{n_B+n_S}\right)\frac{n_S}{2n_S+n_B} < \left(1 - \frac{1}{n_B+n_S}\right)\frac{n_B}{2n_B+n_S} = \lim_{\alpha\to\infty} \pi_S(\alpha)$. Since π_B and π_S are both continuous, there is a unique $\hat{\alpha}(n_B, n_S) > 0$ such that $\pi_B(\hat{\alpha}(n_B, n_S)) = \pi_S(\hat{\alpha}(n_B, n_S))$. That $\hat{\alpha}(n_B, n_S) > 1$ follows from Proposition 1.[17]

To establish part 4, note that:

$$\frac{\partial \pi}{\partial r} = \frac{1}{n_X(1+r)^2(2+1/r)} + \left(1 - \frac{1}{n_X(1+r)}\right)\frac{1}{(2+1/r)^2 r^2} > 0.$$

Since r is decreasing in n_Y if $\alpha < 1$, is constant if $\alpha = 1$ and is increasing if $\alpha > 1$, it follows that the payoff is decreasing, constant, or increasing in the other team's size if h is concave, linear or convex, respectively. To see how a member's payoff is affected by changes in his own team's size, note that:

[17] Recall that in Proposition 1 we indirectly showed that when the individual performance function is linear, a member of the bigger team has a higher payoff.

$$\begin{aligned}\frac{\mathrm{d}\pi}{\mathrm{d}n_X} &= \frac{\partial \pi}{\partial r}\frac{\partial r}{\partial n_X} + \frac{\partial \pi}{\partial n_X} \\ &= \frac{1}{(1+r)(2+1/r)n_X^2}\Big[1 + (1/\alpha - 1)\Big(\frac{r}{1+r} + n_X\frac{1+r}{1+2r} - \frac{1}{1+2r}\Big)\Big].\end{aligned}$$

Clearly, $\mathrm{d}\pi/\mathrm{d}n_X > 0$ if $\alpha \leq 1$. Suppose $\alpha > 1$ and note that $\frac{r}{1+r} \in (0,1)$, $\frac{1+r}{1+2r} \in (0.5, 1)$ and $-\frac{1}{1+2r} \in (-1, 0)$. Thus,

$$\frac{r}{1+r} + n_X\frac{1+r}{1+2r} - \frac{1}{1+2r} \in (0.5n_X - 1, n_X + 1).$$

It follows that $\mathrm{d}\pi/\mathrm{d}n_X > 0$ if $1 + (1/\alpha - 1)(n_X + 1) > 0$ regardless of r (and hence n_Y) or equivalently when $n_X < \frac{1}{\alpha-1}$. Likewise $\mathrm{d}\pi/\mathrm{d}n_X < 0$ if $1 + (1/\alpha - 1)(0.5n_X - 1) < 0$ or equivalently if $n_X > \frac{4\alpha-2}{\alpha-1}$. □

The comparative static results for payoffs allow us to determine when a new player (so that now there are $n_B + n_S + 1$ players in total) would prefer to join the bigger (smaller) team, and when the bigger (smaller) team would like to recruit an additional player (assuming the objective is to maximize the individual member's equilibrium expected payoff).

Proposition 6.
1. Both teams want to recruit an additional player if $\alpha \leq 1$, *while a team with size less than* $\underline{n}(\alpha)$ *prefers to do so whereas a team with size more than* $\overline{n}(\alpha)$ *does not if* $\alpha > 1$.

2. There exists a threshold $\overline{a}(n_B, n_S) > 1$ *such that an additional player prefers to join the bigger team if and only if* $\alpha < \overline{a}(n_B, n_S)$.[18]

Proof. Part 1 is an immediate corollary of Proposition 5(4). To establish part 2, let $\pi^{B+1}(\alpha)$ and $\pi^{S+1}(\alpha)$ denote the additional player's payoff after he joins team B or S, respectively. By Proposition 5(1), π^{B+1} is strictly decreasing in α and π^{S+1} is weakly increasing in α since $n_S + 1 \leq n_B$. Thus, π^{B+1} and π^{S+1} have at most one intersection. Moreover, if $n_S + 1 < n_B$, then we have $\lim_{\alpha\to 0}\pi^{B+1}(\alpha) = 1/2 > 0 = \lim_{\alpha\to 0}\pi^{S+1}(\alpha)$ and $\lim_{\alpha\to\infty}\pi^{B+1}(\alpha) = \Big(1 - \frac{1}{n_B+n_S+1}\Big)\frac{n_S}{2n_S+n_B+1} < \Big(1 - \frac{1}{n_B+n_S+1}\Big)\frac{n_B}{2n_B+n_S+2} = \lim_{\alpha\to\infty}\pi^{S+1}(\alpha)$. Since π^{B+1} and π^{S+1} are both continuous, they have exactly one intersection at some $\overline{a}(n_B, n_S)$ to the left of which π^{B+1} is larger and to the right of which π^{S+1} is larger. If $n_S + 1 = n_B$, then $\pi^{S+1}(\alpha) = \frac{1}{3}(1 - \frac{1}{2(n_S+1)})$ and by a similar argument the same result holds. Since $\pi^{B+1}(1) = \frac{1}{3}\Big(1 - \frac{1}{2n_B+2}\Big) > \frac{1}{3}\Big(1 - \frac{1}{2n_S+2}\Big) = \pi^{S+1}(1)$, we have $\overline{a}(n_B, n_S) > 1$. □

To conclude, we note that the representative case analyzed here — in which total team effort is additively separable, individual performance is a power function of effort and team value is uniformly distributed — is particularly useful for running lab experiments on team contests. Hence, the characterization and comparative statics results provide a helpful benchmark for understanding the actual behavior of subjects in these contests.

[18] Note that this is a different threshold from that in Proposition 5(3). This is because the additional player faces a choice between being a member of a team of size $n_S + 1$ competing with a team of size n_B, versus being a member of a team of size $n_B + 1$ competing with a team of size n_S.

6. Concluding remarks

We proposed to model competition between teams as a contest between two groups of players in which each individual player incurs the cost of his own effort and the team's overall performance is some aggregation of the individual efforts of its members. What makes a collection of individuals a "team" is the fact that the prize for winning is a pure public good for its members whose value is common knowledge among them. In contrast, the value of the prize to the opposing team is not observed and is treated as a random variable.

The model allowed us to determine whether the bigger team has an advantage, and whether its members are better off than those of the smaller team. The results shed new light on the "group-size paradox" by showing that the strategic effect of size depends on the shape of the individual performance function. Thus, the size advantage may depend on the particular task at hand, which will determine how the marginal contribution of effort changes with the level of effort.

Future research should explore how other characteristics — such as the composition of heterogenous teams or the communication protocols among members — affect the outcome of competition. The ultimate goal is to incorporate into standard models of strategic interaction the idea that the players are part of groups of individuals who have to consider the actions of their teammates, as well as those of the opposing teams.

Appendix A

A.1. General team performance function

We generalize the model in the main text. Team X's overall performance is now given by the general team performance function $H_{n_X} : \mathbb{R}_+^{n_X} \to \mathbb{R}_+$ where $H_{n_X}(e_1, ..., e_{n_X})$ is team X's performance given efforts $e_1, ..., e_{n_X}$ from its members. Note that the team performance function only depends on a team's size. For any team size n, we impose the following assumptions on H_n:

A1 (Monotonicity) H_n is strictly increasing in each argument.
A2 (Differentiability) H_n is twice differentiable in each argument.
A3 (Symmetry) If $(e_1, ..., e_n)$ is a permutation of $(e'_1, ..., e'_n)$, then $H_n(e_1, ..., e_n) = H_n(e'_1, ..., e'_n)$.

Moreover, for any two team sizes N and n where $N > n$, we assume the following:

A4 $H_N(\overbrace{0, ..., 0}^{N \text{ copies}}) = H_n(\overbrace{0, ..., 0}^{n \text{ copies}}) = 0.$
A5 $H_N(\overbrace{e, ..., e}^{N \text{ copies}}) > H_n(\overbrace{e, ..., e}^{n \text{ copies}})$ for any $e > 0$.

A5 says that if members of both teams put in the same positive effort e, then the bigger team performs better. A5 implies that the bigger team has an a priory size advantage.

It is straightforward to verify that the additive separable case in the main text where $H_n(e_1, ..., e_n) = \sum_{i=1}^n h(e_i)$ satisfies A1–A5.

Since H_n is symmetric, its partial derivative with respect to any argument is the same when evaluated at $(e, ..., e)$. Denote this partial derivative by $\eta_n(e)$. Also, denote $H_n(e) \equiv H_n(e, ..., e)$.

Lemmas 1 and 2 both hold for general team performance functions using essentially the same proofs. We establish a result that implies Theorem 1 as a corollary.

Theorem 2. *In the in-team symmetric monotone PBNE:*

1. $P_B(v) > P_S(v)$ *for any* $v \in (0, 1)$ *and hence team B is more likely to win, if* $\eta_{n_B}(e) > \eta_{n_S}(\hat{e})$ *for any* $e, \hat{e} \in (0, 1]$ *where* $H_{n_B}(e) = H_{n_S}(\hat{e})$.
2. $P_B(v) < P_S(v)$ *for any* $v \in (0, 1)$ *and hence team S is more likely to win, if* $\eta_{n_B}(e) < \eta_{n_S}(\hat{e})$ *for any* $e, \hat{e} \in (0, 1]$ *where* $H_{n_B}(e) = H_{n_S}(\hat{e})$.
3. $P_B(v) = P_S(v)$ *for any* $v \in [0, 1]$ *and hence both teams win with the same probability, if* $\eta_{n_B}(e) = \eta_{n_S}(e) = \gamma$ *for any* $e \in (0, 1]$, *where* γ *is a constant.*

Proof. Observe that for any $p \in [0, P_X(1)]$ there is a unique $\epsilon_X(p)$ such that $p = H_{n_X}(\epsilon_X(p))$. It follows that $e_X(v) = \epsilon_X(P_X(v))$.

We first show parts 1 and 2. Consider the in-team symmetric monotone PBNE. Fix any v such that $P_B(v) = P_S(v) > 0$. Part 1 of Lemma 1 implies that such a v exists. Define $t \equiv P_B(v) = P_S(v)$. Thus, first-order conditions are given by:

$$f(v)(P_S^{-1})'(t)\eta_{n_B}(\epsilon_B(t))v = 1$$
$$f(v)(P_B^{-1})'(t)\eta_{n_S}(\epsilon_S(t))v = 1.$$

It follows that:

$$\frac{\eta_{n_B}(\epsilon_B(t))}{\eta_{n_S}(\epsilon_S(t))} = \frac{(P_B^{-1})'(t)}{(P_S^{-1})'(t)}. \tag{A.1}$$

Obviously, no player would exert effort greater than 1 in equilibrium. Note that $H_{n_B}(\epsilon_B(t)) = H_{n_S}(\epsilon_S(t)) = t$. Thus, if $\eta_{n_B}(e) > \eta_{n_S}(\hat{e})$ for any $e, \hat{e} \in (0, 1]$ where $H_{n_B}(e) = H_{n_S}(\hat{e})$, then the left-hand side of equation (A.1) is greater than 1 for any $v \in (0, 1]$ where $P_B(v) = P_S(v) = t$, implying $(P_B^{-1})'(t) > (P_S^{-1})'(t)$ or equivalently $P_B'(v) < P_S'(v)$. Thus, whenever the curves of P_B and P_S intersect when P_B and P_S are strictly positive, the curve of P_S is steeper than the curve of P_B, implying that they intersect at most once in the strictly positive region since part 2 of Lemma 1 implies P_B and P_S are continuous. Part 1 of Lemma 1 implies that they intersect at $\overline{P}$ (when $v = 1$). Thus, $P_B(v) > P_S(v)$ for any $v < 1$ where $P_B(v) > 0$. Combined with part 3 of Lemma 1, this implies that $P_B(v) > P_S(v)$ for any $v \in (0, 1)$. Therefore, team B is more likely to win. Part 2 is proven with a symmetric argument.

To establish part 3, suppose that $\eta_{n_B}(e) = \eta_{n_S}(e) = \gamma$, where γ is a constant, for any e. The first-order condition becomes $G_Y'(P_X(v))\gamma v = 1$. The boundary value problem characterizing the in-team symmetric monotone PBNE does not depend on team sizes or H_{n_X} and H_{n_Y}, and hence neither does the solution. Part 3 follows from symmetry and uniqueness. □

The above sufficient condition is fairly intuitive. If the two teams tie in performance when all members of the same team choose the same effort, the team that is more likely to win is the one in which a member is always marginally more productive.

To verify that Theorem 1 is implied by Theorem 2, note that in the additively separable case we have $H_n(e) = nh(e)$ and $\eta_n(e) = h'(e)$. Pick any $e > 0$ and $\hat{e} > 0$ where $n_B h(e) = n_S h(\hat{e})$. We have $e < \hat{e}$ because h is strictly increasing. If h is strictly concave, then $\eta_{n_B}(e) > \eta_{n_S}(\hat{e})$ and by part 1 of Theorem 2 the bigger team is more likely to win. The convex and linear cases are analogously established.

A.2. *On Remark 1*

A PBNE is said to satisfy *active participation* if there is no player who always exerts zero effort regardless of the valuation.

Proposition 7.

1. *If h is strictly concave, then there is a unique monotone PBNE, which is the in-team symmetric one.*
2. *If h is strictly convex, then there is a unique active participation monotone PBNE, which is the in-team symmetric one.*
3. *If h is linear, then there is a continuum of monotone PBNE, all of which have the same team performance as the in-team symmetric one.*

Proof. Let $(e_{X,i})_{X\in\{B,S\}}, i \in \{1, ..., n_X\}$ be a (not necessarily in-team symmetric) monotone PBNE where $e_{X,i}(v)$ is the effort exerted by player i on team X given valuation v. Lemma 1 holds for this equilibrium as well since in-team symmetry is not required for its proof. For any $v \in (0, 1)$, the first-order condition faced by player i on team X is thus:

$$G'(P_X(v))h'(e_{X,i}(v))v = 1. \tag{A.2}$$

Suppose h is strictly concave or convex. Pick any two players i and j on team X. If $e_{X,i}(v) > 0$ and $e_{X,j}(v) > 0$, then first-order condition (A.2) implies that $h'(e_{X,i}(v)) = h'(e_{X,j}(v))$, which in turn implies $e_{X,i}(v) = e_{X,j}(v)$ since h' is strictly monotone. Thus, if h is strictly concave or convex, then in any monotone PBNE, given any v, two players on the same team exert the same effort if neither shirks.

To show part 1, suppose h is strictly concave. For team X, choose any v such that $e_{X,i}(v) > 0$ for some i. Thus, first-order condition (A.2) holds for i. If there is some j such that $e_{X,j}(v) = 0$ then since $h'(0) > h'(e_{X,i}(v))$ we have:

$$G'_Y(P_X(v))h'(0)v > 1,$$

implying that player j can profit by increasing his effort, a contradiction. Thus, there does not exist any v such that on the same team some players exert effort while others do not, which along with the result from the previous paragraph implies that any monotone PBNE must be in-team symmetric.

To show part 2, suppose that h is strictly convex. Clearly, the in-team symmetric monotone PBNE satisfies active participation. We now show that an active-participation monotone PBNE must be in-team symmetric. Pick any active-participation monotone PBNE. Suppose there are players i and j on team X whose effort functions differ one from the other. Thus, there exists some v such that $e_{X,i}(v) \neq e_{X,j}(v)$. Without loss of generality, assume $e_{X,i}(v) > e_{X,j}(v)$. It follows that $e_{X,j}(v) = 0$ by the result from the first paragraph of this proof. Since $e_{X,j}$ is weakly increasing and not constant, there is some $\overline{v} \geq v$ such that $e_{X,j}(\overline{v} - \epsilon) = 0$ and $e_{X,j}(\overline{v} + \epsilon) > 0$ for any $\epsilon > 0$ if $\overline{v} < 1$, or $e_{X,j}(\overline{v} - \epsilon) = 0$ and $e_{X,j}(\overline{v}) > 0$ for any $\epsilon > 0$ if $\overline{v} = 1$. Suppose $\overline{v} < 1$. We then have:

$$e_{X,j}(\overline{v} + \epsilon) = e_{X,i}(\overline{v} + \epsilon) \geq e_{X,i}(v) > 0.$$

It follows that:

$$\begin{aligned}\lim_{\epsilon\to 0}\Big(P_X(\overline{v}+\epsilon)-P_X(\overline{v}-\epsilon)\Big) &\geq \lim_{\epsilon\to 0}\Big(h(e_{X,j}(\overline{v}+\epsilon))-h(e_{X,j}(\overline{v}-\epsilon))\Big)\\ &\geq h(e_{X,i}(v))-h(0)\\ &= h(e_{X,i}(v))\\ &> 0.\end{aligned}$$

However, this is a contradiction since P_X is continuous at $\overline{v}$ as implied by Lemma 1. The fact that $\overline{v}=1$ also leads to a similar contradiction. Thus, $e_{X,i}(v)=e_{X,j}(v)$ for any v. This establishes part 2.

Finally, we show part 3. If h is linear then h' is some constant $\gamma>0$. First order condition (A.2) does not depend on X and i. Hence any weakly increasing effort functions $(e_{X,i})_{X\in\{B,S\}}, i\in\{1,...,n_X\}$ where $\sum_{i=1:n_X} e_{X,i}(v)=P_X(v)$, $X\in\{B,S\}$ constitute a monotone PBNE as long as (P_B,P_S) solve the boundary value problem given by first-order condition (A.2), $P_B(1)=P_S(1)$ and $\min\{P_B^{-1}(0),P_S^{-1}(0)\}=0$. The proof of Theorem 1 implies that the solution is unique. Part 3 follows. □

It is worth noting that if h is convex, then there are additional monotone PBNE with *non-active* participation. It is easy to verify that each of these equilibria looks exactly like the unique active-participation PBNE of the reduced competition when all the shirking members are removed.

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2018.04.006.

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